

# Application-inspired learning of mathematics for students of electrical engineering with the aid of GeoGebra as an information technology tool

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*Abstract*— In this paper we present the novel approach to the application-inspired learning, by introducing GeoGebra as an information technology tool for mathematical education of students of electrical engineering and computer science. GeoGebra applets developed in the concept of application-inspired learning do not require prior knowledge of the information technology students' and provide lecturers with the ability to customize the presentation according to prepared guidance of students in the learning process.

## I. INTRODUCTION

During our careers in teaching mathematics to students of electrical engineering and computer science, we have observed that their interest for the mathematical concepts being taught significantly increases if they are illustrated with practical examples from physics or engineering. Furthermore, the concept of application-inspired learning is showing the increase in students' self-efficiency and positive feeling towards mathematics [4]. Motivated by this work, we are developing several modules that start with the concrete challenging application, use creative exploration of the problem by GeoGebra as an information technology tool, examining solution paths, and then forging the mathematical tools. In such approach the application is not merely the illustration of certain mathematical tool, but represents a problem that the students are solving and on the way discovering the appropriate mathematical tool.

## II. GEOGEBRA

We investigate how the applied problem can be explored by students using GeoGebra and increase the efficiency of the application-inspired learning concept. GeoGebra is a dynamic mathematical environment in which all representations of mathematical objects are linked and allow students to view, experiment with, and analyze problems and situations in a laboratory like setting [1]. The crucial part of this process is to create a valuable set of assignments and questions to be addressed by the students, guided by the lecturers, before the mathematical theory is presented. GeoGebra provides excellent tools for students to explore different aspects of

the applied problem presented to them, thus increasing the positive effects of the application-inspired learning. GeoGebra is an interactive free software program for learning and teaching mathematics from primary school up to university level. Depending on the course we were lecturing and the level of knowledge of our students, we were just presenting already prepared GeoGebra applets to illustrate same important theoretic concepts or we were training our students to use or to create them. The best results are achieved when same topic is being upgraded through different math courses. In our work we present one of the modules that we are developing to be implemented in the classes of mathematics for students of electrical engineering, with the aim to increase the quality of the learning process by motivating students and connecting topics relevant to them with new mathematical concepts. There are different approaches depending of the study level. We chose to create GeoGebra applets in order to give students opportunity to interactively observe how the changes in some parameters are changing outputs.

## III. DIFFERENTIAL EQUATIONS IN GEOGEBRA

We illustrate the concept of application-inspired learning using GeoGebra as information technology tool on the examples of the motion of a body along a straight line (boat problem) and damped free oscillations, for learning the concepts of differential equations on the first year of studies.

Let us consider the following problem. A sailing boat of mass  $m$  moves through water. At time  $t = 0$  it has speed  $v_0$ . If the resisting force is proportional to the velocity with the proportional factor  $k$ , calculate the boat velocity as a function of time. By Newton's Law of motion, the total force on the body is

$$F_n = m v'(t), \quad (1)$$

where  $v'(t)$  is the acceleration of the mass. Since the resisting force is proportional to the velocity with the proportional factor  $k$ , we have

$$F_r = k v(t). \quad (2)$$

Net force is equal to minus resisting force, so we get

$$F_n = -F_r, \quad (3)$$

i.e. we obtain homogeneous linear differential equation of the first order

$$m v'(t) = -k v(t). \quad (4)$$

The general solution of the boat problem is given by

$$v(t) = C e^{-\frac{k}{m}t}. \quad (5)$$

If we consider the differential equation (5) with initial conditions  $v(0) = v_0$  we obtain

$$v(t) = v_0 e^{-\frac{k}{m}t}. \quad (6)$$

Now, we will give the detailed explanation of the GeoGebra applet presented on the Fig. 1. Firstly, we create three sliders for mass  $m$ , resisting constant  $k$  and time  $t$ . Slider for time (not visible in Fig. 1.) is necessary for the animation. Insert Buttons “Start” and “Stop” are used to begin and terminate animation. These tools are used to execute `StartAnimation[t]` and `StartAnimation[false]` GeoGebra script commands with a single click on buttons. The **Input Box** “ $v_0 =$ ” is used to change interactively initial condition and it is connected to object  $v_0$ . Changing the value of the parameter  $m$  changes the size of the boat. We have done it by using transform tool *Dilate from point*, “`slikal=Dilate(slikal, e, F)`”, where `slikal` is image of the boat,  $e$  is ratio of homothety proportional to  $m$ , and  $F$  is a center point from which all other points are either enlarged or compressed. Changing the value of the parameter  $k$  changes the speed of boat motion. And finally boat motion is realized through time slider by transform tool *Translate by vector*, “`slikal=Translate(slikal, u)`”, where  $u$  is displacement vector defined by starting point  $F$  and point  $G$ . Since we just need the horizontal displacement the first coordinate of the point  $G$  is variable  $t$  and the second one is fixed.

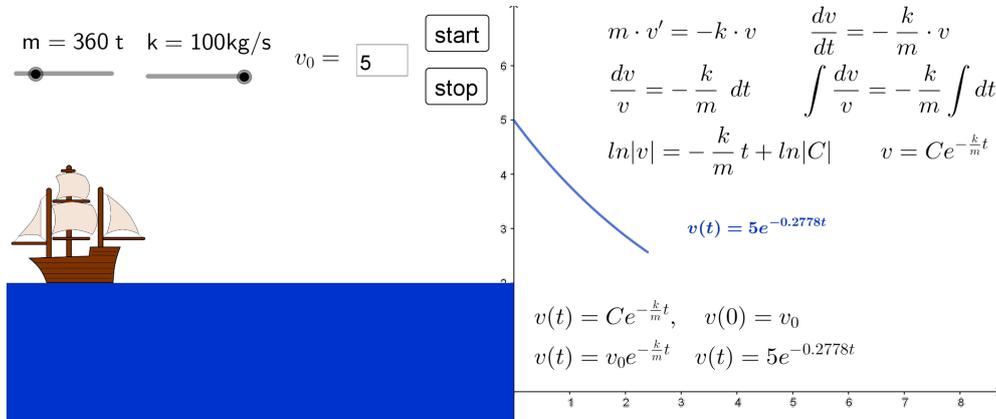


Figure 1. Boat problem

Let's take a look on Algebraic View of our applet.

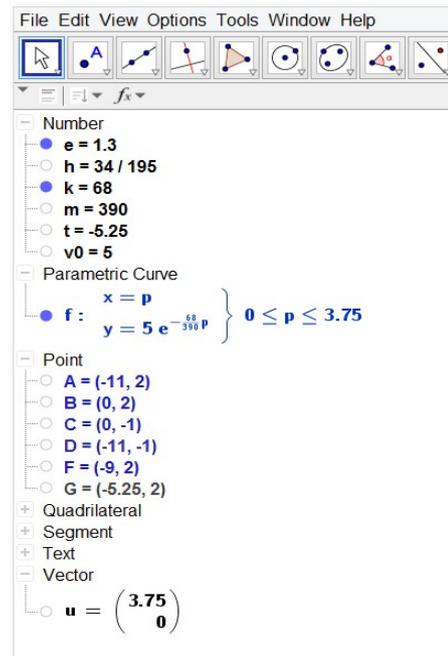


Figure 2. Algebraic View 1

Numbers  $h$  is defined as  $h = \frac{k}{m}$ , and we use it to define parametric curve  $f$ . Points  $A, B, C$  and  $D$  are corners of the blue rectangle which represents the sea.

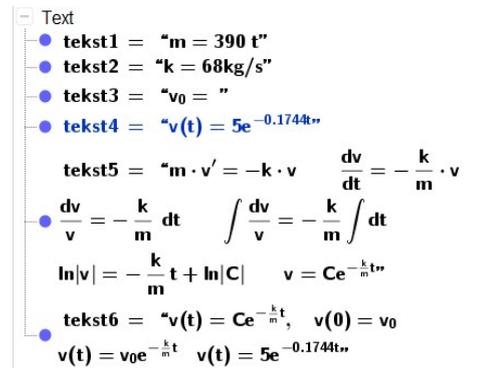


Figure 3. Algebraic View 2

Figure 3. deals with printing text with interactive object on the right side of applet.

Our second example is mass-spring-damper system. Let us consider a mechanical system consisting of a body of mass  $m$ , hanging on the elastic wire with the stiffness coefficient  $k$ , damped in the viscous fluid with the viscous damping coefficient  $b$ . The equation describing such damped free oscillation, based on the Newton's law, is

$$m y''(t) + b y'(t) + k y(t) = 0, \quad (7)$$

with initial conditions

$$y(0) = y_0, y'(0) = y_1. \quad (8)$$

Here  $y(t)$  is time varying displacement,  $y'(t)$  is velocity and  $y''(t)$  is acceleration. We have prepared for the students a Geogebra applet enabling them to visualize the displacement  $y(t)$ , both the graphical and analytical form, as well as the actual simulation of the damped free oscillation, for different values of  $m, k, b$ . The values of  $m, k, b$  in the applet belong to a certain range, chosen so that we only consider the case of under-damping system. We can assume that solution of differential equation (7) is in the form of exponential function  $y(t) = e^{\lambda t}$ . This leads to the equation

$$m\lambda^2 + b\lambda + k = 0, \quad (9)$$

which has the solutions  $\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$ . In our motivational example we only consider the case when  $b^2 - 4mk < 0$ . By applying the properties of complex function, we get that the displacement is of the form

$$y(t) = e^{-\frac{b}{2m}t} (C_1 \cos(\omega t) + C_2 \sin(\omega t)), \quad (10)$$

where  $\omega = \frac{\sqrt{4mk - b^2}}{2m}$  is damped circular frequency. The solution can be rewritten in the form

$$y(t) = A e^{-\frac{b}{2m}t} \sin(\omega t + \varphi), \quad (11)$$

where  $A = \sqrt{C_1^2 + C_2^2}$  and  $\varphi = \arctan \frac{C_1}{C_2}$ . Initial

conditions (8) implies that  $A = 2\sqrt{\frac{m(ky_0^2 + by_0y_1 + my_1^2)}{4mk - b^2}}$

and  $\varphi = \arctan \left( \frac{y_0 \sqrt{4mk - b^2}}{by_0 + 2my_1} \right)$ .

The construction of the GeoGebra applet is similar to the first one and all details can be found in [1].

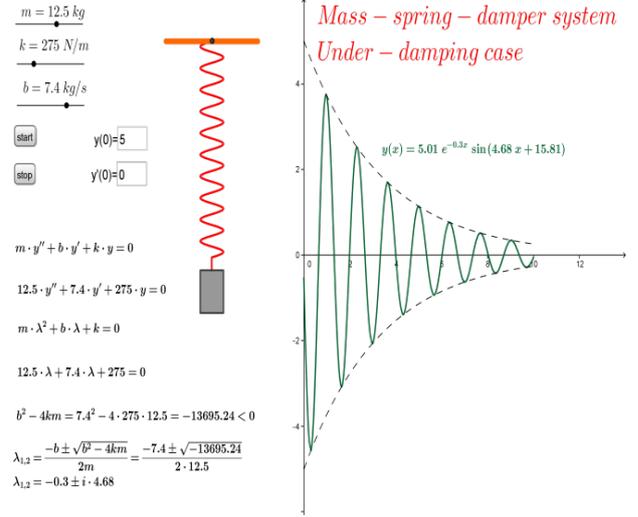


Figure 4. Mass-spring-damper system

#### IV. SAME PROBLEM FOR DIFFERENT LEVELS OF STUDIES

We will concentrate a bit more on the mass-spring-damper system, because it nicely presents different mathematical concepts for solving the same problem. On the high-school level, problem is solved using linear equation. On the under graduate levels, the second order differential equation with constant coefficient with initial conditions is more suitable, since students are better equipped with appropriate mathematical knowledge. On the first year of under graduate studies applets of this type are just presented as a part of the lectures. But on the second-year students are encouraged to play with these applets and to form similar ones. Finally, on the graduate level of studies we can present this model as control linear system. For its solving we can use either transform function derived by Laplace transform or Jordan form and generalized eigenvectors.

Linear time invariant control system is a system of the form

$$\begin{aligned} \dot{\vec{x}}(t) &= A\vec{x}(t) + B\vec{u}(t) \\ \vec{y}(t) &= C\vec{x}(t) + D\vec{u}(t), \end{aligned} \quad (12)$$

where  $\vec{x}(t) = [x_1(t) x_2(t) \dots x_n(t)]^T$  is a state vector,  $\vec{u}(t) = [u_1(t) u_2(t) \dots u_m(t)]^T$  is input vector and  $\vec{y}(t) = [y_1(t) y_2(t) \dots y_r(t)]^T$  is output vector. Matrix  $A$  is called state matrix, matrix  $B$  input matrix,  $C$  is output matrix and  $D$  is direct link. In most systems  $D = 0$ , [2], [5].

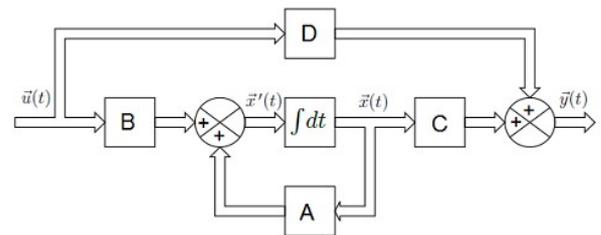


Figure 5. LTICS

Consider again mass-spring-damper system. It consists of a weight  $m$  on a spring with spring constant  $k$ , its motion damped by friction with coefficient  $b$ . If  $y(t)$  is displacement from the resting position and  $u(t)$  is the force applied, we have

$$m y''(t) + b y'(t) + k y(t) = u(t). \quad (13)$$

Let  $x_1(t) = y(t)$  and  $x_2(t) = y'(t)$  be new variables, called state variables. Then the previous equation (13) is equivalent to the system

$$\begin{aligned} x_1'(t) &= x_2(t) \\ x_2'(t) &= -\frac{k}{m} x_1(t) - \frac{b}{m} x_2(t). \end{aligned} \quad (14)$$

Since  $y(t)$  is of interest, the output equation  $y(t) = x_1(t)$  is also added. We can now rewrite system as

$$\begin{aligned} \begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u(t), \\ y(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}. \end{aligned} \quad (15)$$

Using knowledge of Jordan form, eigenvalues and eigenvectors we obtain solution of given Cauchy problem

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = S \cdot e^{tJ} \cdot S^{-1} \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} + \int_0^t S \cdot e^{(t-s)J} \cdot S^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(s) ds.$$

Matrix  $J$  is Jordan canonical form of the system matrix

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}. \text{ For matrix } S, A = S \cdot J \cdot S^{-1} \text{ holds, i. e.}$$

the columns of matrix  $S$  are the generalized eigenvectors of the matrix  $A$ . Since  $y(t) = x_1(t)$  we easily get the value of output parameter  $y(t)$  depending on the input parameter  $u(t)$ .

The transfer function of the linear time invariant system is the ratio of Laplace transform of the output  $Y(s)$  to the Laplace transform of the corresponding input  $U(s)$  with all initial conditions assumed to be zero.

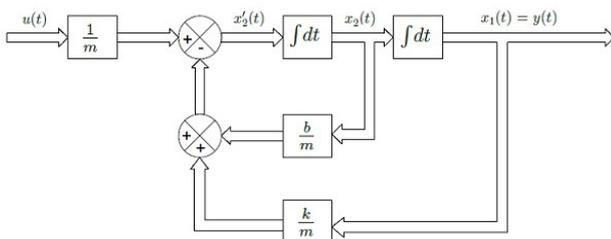


Figure 6. Mass-spring-damper system LTIS shema

Taking Laplace transform of the both sides of the equation (13) with initial conditions  $y(0) = y_0$  and  $y'(0) = y_1$  we obtain

$$m(s^2 Y(s) - s y_0 - y_1) + b(s Y(s) - y_0) + k Y(s) = U(s) \quad (16)$$

where

$$Y(s) = L\{y(t)\} = \int_0^{\infty} e^{-st} y(t) dt \quad (17)$$

and

$$U(s) = L\{u(t)\} = \int_0^{\infty} e^{-st} u(t) dt. \quad (18)$$

Combining terms with respect to  $Y(s)$  we obtain

$$(m s^2 + b s + k) Y(s) = U(s) + (m s + b) y_0 + m y_1. \quad (19)$$

The Laplace transform for the system assuming zero initial conditions is

$$(m s^2 + b s + k) Y(s) = U(s). \quad (20)$$

Transfer function from force input to displacement output is

$$(m s^2 + b s + k) Y(s) = U(s). \quad (21)$$

## V. SOME SET UP QUESTIONS FOR STUDENTS RELATED TO THE MASS-SPRING-DAMPER SYSTEMS

The following are some of the questions raised, answered by the students using GeoGebra, with the guidance by the lecturers, when needed.

- Fix two of the values mass  $m$ , spring constant  $k$  or viscous damping coefficient  $b$  and observe how the displacement  $y(t)$  changes as the other one changes its value in the given range.
- Notice that we have sinusoidal motion decreasing in amplitude with time. How does the exponential factor in the analytical form of the displacement  $y(t)$  contribute to the amplitude decrease?
- Measure the time at which  $y(t)$  achieves its maximum (when the derivative vanishes). The successive zeros of the derivative  $y'(t)$  are separated by a time lapse of  $\frac{\pi}{\omega}$ . Why? Prove that for two neighboring maximum points  $t_1$  and  $t_2$  we have that  $t_1 - t_2 = \frac{2\pi}{\omega}$ . Explain in details

how damped circular frequency can be calculated using velocity  $y'(t)$ .

- Two characteristics of under-damped free oscillation are natural circular frequency

$$\omega_n = \sqrt{\frac{k}{m}} \quad \text{and damping ratio } \zeta = \frac{b}{2m\omega_n}. \text{ We}$$

can rewrite the equation (7) in this new fashion as

$$y''(t) + 2\zeta\omega_n y'(t) + \omega_n^2 y(t) = 0. \quad (22)$$

Roots of the characteristic equation

$$\lambda^2 + 2\zeta\omega_n \lambda + \omega_n^2 = 0 \quad (23)$$

are

$$\lambda_{1,2} = \omega_n \left( -\zeta \pm i\sqrt{1-\zeta^2} \right) = -\zeta\omega_n \pm i\omega, \quad (24)$$

where  $\omega = \omega_n \sqrt{1-\zeta^2}$  is the damped circular frequency.

- Measure the ratio of the value of the displacement at two successive maximum. The difference of their natural logarithms is defined as

$$\Delta = \ln y(t_2) - \ln(t_1) = \ln \frac{y(t_2)}{y(t_1)} \quad (25)$$

and called the logarithmic decrement. Therefore, we have that  $y(t_2) = y(t_1)e^\Delta$ . Furthermore, values of  $\sin(\omega t + \varphi)$  at two points of time

differing by  $\frac{2\pi}{\omega}$  are equal, so we get

$$\frac{y(t_2)}{y(t_1)} = \frac{e^{-\zeta\omega_n t_1}}{e^{-\zeta\omega_n t_2}} = e^{\zeta\omega_n(t_2-t_1)} = e^{\frac{2\pi\zeta\omega_n}{\omega}}. \quad (26)$$

- Express the natural circular frequency and the damping ratio in function of the logarithmic decrement and the damped circular frequency.

## VI. CONCLUSION

Inspired by [4], we included GeoGebra in the application-inspired learning as an information technology tool which allows us to have a better control of the process, and the students to conveniently explore the applied problem without the need of prior knowledge of the information tool used. GeoGebra applets provide the ability to show different aspects of the presented setting by turning on the show option for different formulas and connections, according to the analysis needs.

## REFERENCES

- [1] J. Hohenwarter, M. Hohenwarter, *Introduction to GeoGebra*, 2011;
- [2] C.F. Beards, *Engineering Vibration Analysis with Application to Control Systems*, (1995), London: Edward Arnold.
- [3] B. Malešević, I. Jovović, B. Banjac, "Visualization in teaching and learning mathematics in elementary, secondary and higher education", in *Buletinul Științific al Universității „POLITEHNICA“ din Timișoara, Seria HIDROTEHNI TRANSACTIONS on HYDROTECHNICS*, Tom 58(72), Fascicola 1, 2013, pp. 37-40.
- [4] H. Moon, T. Asaki, M. Snipes, C. Camfield, J. Frost, "Data-Driven Applications Inspiring Linear Algebra", in *SIAM Conference on Applied Mathematics Education*, [September 30 – October 2, 2016 in Philadelphia, Pennsylvania, USA].
- [5] K. Ogata, *Modern Control Engineering*, Prentice Hall, 2002, New Jersey.